Intersections of t-reguli, Rational Curves, and Orthogonal Latin Squares

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ABSTRACT

The celebrated net-embedding theorem of R. H. Bruck asserts that a net with a large number of parallel classes, i.e., a net with small deficiency, can be embedded in an affine plane. The only known class of examples of unembeddable nets of small deficiency has been constructed by geometry, using partial spreads of PG(3, q). In this note we attempt to generalize that discussion to PG(n, q), with particular emphasis on the case $n = 5$.

0. INTRODUCTION, BACKGROUND

The central problem in the theory of finite planes is, still, the one of existence. Related to this is the issue of when a system of mutually orthogonal latin squares (m.o.l.s.) of order n can be embedded in a complete such system of m.o.l.s. of order n. A celebrated result of R. H. Bruck is that a "large" system of m.o.l.s. of order n (i.e., one in which the deficiency is of the order of $n^{1/4}$) can be embedded in a complete system. As a counterpoint to this result of Bruck we exhibited in [4, 5] an unimbeddable system of m.o.l.s. of order n having deficiency $n^{1/2}$. In fact that system was shown to be maximal. For the system in question $n = q^2$, with q a prime power $q = p^r$. The method in [4, 5] makes use of a geometrical construction involving translation nets coming from partial spreads in PG(3, q). Although the analogous construction can easily be carried out in the higher dimensions, a proof that it yields a maximal

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net, or even a maximal partial spread, seems to be difficult. However, by a "fluke of nature," which seems to be of interest in its own right, we are able to make some progress for the case of partial spreads in $\mathrm{PG}(5, q)$. Let us explain as follows. In $\Lambda = \mathrm{PG}(2t+1, q)$, let R_t be a t-regulus, and let Σ be a $(t+1)$-dimensional subspace of Λ. Each axis space of R_t must then meet Σ in at least a point. In Section 1 we examine the case when each axis space of R_t intersects Σ in *precisely* one point. We show in Theorem 1.2 that the $q+1$ points so obtained lie on a rational curve C. It was of interest to us to determine that the cases $t=2$ and $t>2$ are very different, as follows. If $t=2$, that is, if $\Lambda = \mathrm{PG}(5, q)$, then C must be a *normal* rational curve in Σ. This result, which is of crucial importance in the sequel, follows from a counting argument of Freeman [12, Proposition 10]. However, if $t>2$, then we show in Theorem 1.4 that, in general, C is far from being normal in Σ.

In subsequent sections we exploit this behavior of C when $t=2$ in connection with partial spreads, switching sets, nets and, accordingly, with orthogonal latin squares. Maximal partial 1-spreads are studied in several papers [1, 4, 6, 9], and maximal partial t-spreads in [7]. Mesner in [14] and Bruen in [4, 6] have obtained bounds on the cardinality of a maximal partial 1-spread. (Using an interesting idea, Ebert [10] has improved the lower bounds.) Bruen [6, 7] has generalized these bounds to the case of maximal partial t-spreads with $t \geqslant 1$. In Theorem 2.2, using an idea in [4] and the result in [12], we show that the upper bounds in [7] for maximal partial 2-spreads are reasonably good: namely we exhibit a "large" maximal partial 2-spread W in $\mathrm{PG}(5, q)$. This construction generalizes the construction of a partial 1-spread W in [4, 5]. However, curiously, in [4] a nonregular spread is needed for the construction, whereas here we use a regular spread of $\mathrm{PG}(5, q)$ in order to construct W.

Any partial t-spread K in Λ of cardinality λ yields a certain kind of net, called a *translation net* $N = N(K)$, having exactly λ parallel classes, and having order q^{t+1}. N then yields a set of $\lambda - 2$ mutually orthogonal latin squares of size $q^{t+1} \times q^{t+1}$. One approach to the existence question for finite planes has been to try to embed nets in planes. In [5] it was shown that the Bruck bound described earlier is close to being best possible by demonstrating that, in certain cases, the net $N(W)$ above is not embeddable. For example, in [5] there is exhibited *a maximal set of 20 mutually orthogonal latin squares of order 25*. Accordingly it would be of interest to have more information on the net $N(W)$ of Theorem 2.2. In Section 3 we are able to obtain this information in the case $q=2$. The case $q=2$ is also of particular interest in that it is closely related to a question of Drake [8] regarding the existence of a maximal set of four mutually orthogonal latin squares of order 8.

We come to switching sets. One method of constructing a new affine plane from a given affine plane is to use a replaceable net. In a representation of an affine plane the parallel classes of a replaceable net may sometimes be given by a switching set. In Theorems 2.2 and 3.3 we are able to rule out the existence of certain "geometrical" types of switching sets.

Recall [3, 6] that a *partial t-spread* of $\mathrm{PG}(d,q)$, the projective space of dimension d over $\mathrm{GF}(q)$, is a set W of pairwise disjoint (skew) t-dimensional subspaces of $\mathrm{PG}(d,q)$. If each point of $\mathrm{PG}(d,q)$ is contained in some element of W, then W is called a *t-spread*. A partial t-spread W is called *maximal* provided (i) and (ii) below are satisfied:

(i) W is not a spread.
(ii) W is not properly contained in a larger partial t-spread.

A *t-regulus* in $\mathrm{PG}(2t+1,q)$ is a partial t-spread R_t, of cardinality $q+1$, with the property that if a line meets three elements of R_t, then the line meets all elements of R_t (in a point). An element of R_t is called an *axis space*, and a line meeting all the axis spaces is called a *transversal line*. We can think of R_t as the *Segre product* of a line by a t-dimensional subspace of $\mathrm{PG}(2t+1,q)$ [16, p. 12].

1. INTERSECTIONS OF REGULI

The following result, which gives the possible intersections in $\mathrm{PG}(5,q)$ of a three-dimensional projective space with a 2-regulus, is crucial in the sequel. A proof is contained in [12, Proposition 10, p. 178]. There the assumption was made that q was odd. However, the proof in [10] is valid for all $q=p^t$, p a prime. We have

RESULT 1. *A three-dimensional subspace Σ_3 of $\mathrm{PG}(5,q)$ is in exactly one of the following classes with respect to a 2-regulus R_2 of $\mathrm{PG}(5,q)$:*

I. *Σ_3 contains one axis plane of R_2 and meets each of the remaining axis planes of R_2 in a point. These points lie on a transversal of R_2.*

II. (i) *Σ_3 intersects R_2 in a nondegenerate hyperbolic quadric.*
(ii) *Σ_3 intersects R_2 in three lines, say r,s,t, where r,s belong to distinct axis planes, t is a transversal line of R_2, and t intersects each of r and s in a point.*
(iii) *Σ_3 intersects R_2 in a plane tangent to a nondegenerate hyperbolic quadric contained in R_2.*
(iv) *Σ_3 intersects one axis plane of R_2 in a line s. The remaining q points of intersection are coplanar and lie on a nondegenerate conic such that s meets the plane of this conic in exactly one point. This point is on the conic.*

III. (i) Σ_3 *intersects* R_2 *exactly in a transversal line.*
(ii) Σ_3 *intersects* R_2 *in a set of* $q+1$ *points with no four of them coplanar.*

REMARK. Since we are working in PG$(5,q)$, each axis plane of R_2 must meet Σ_3 in at least a point. Using the classification in result 1, the following may be deduced.

COROLLARY 1.1. *Assume that each of the* $q+1$ *axis planes of* R_2 *intersects* Σ_3 *in just one point. Then the* $q+1$ *points of intersection of* Σ_3 *and* R_2 *either form the points of a transversal line to* R_2 *or have the property that no four of them are coplanar.*

We proceed to examine the situation in higher dimensions. Working in PG$(2t+1,q)$, let $U=\Sigma_{t+1}$ be a $(t+1)$-dimensional subspace and let R_t be a t-regulus. Then, by dimension theory, each element of R_t meets U in at least one point.

THEOREM 1.2. *In* PG$(2t+1,q)$ *suppose that each of the* $q+1$ *axis spaces of a t-regulus* R_t *meets a* $(t+1)$*-dimensional subspace* $U=\Sigma_{t+1}$ *in just one point. Then the* $q+1$ *points of intersection of* Σ_{t+1} *and* R_t *lie on a rational curve C. If C is a line, then C is a transversal to* R_t.

Proof. For PG$(2t+1,q)$ let $V=V(2t+2,q)$ be the underlying vector space. We may write V as the direct sum $V=V_1\oplus V_2$, where V_1, V_2 are $(t+1)$-dimensional subspaces of V. Thus each vector $\mathbf{v}$ in V may be written as a pair, so $\mathbf{v}=(\mathbf{x},\mathbf{y})$, where $\mathbf{x}$, $\mathbf{y}$ are unique vectors in V_1, V_2. The zero vector is denoted by $\mathbf{0}$. Following Tyrrell and Semple [17], we may, without loss of generality, take the axis spaces of R_t to be the space X with equation $\mathbf{x}=\mathbf{0}$ together with a collection of q subspaces T_λ corresponding to the q values of λ in GF(q). Here T_λ is defined by

$$T_\lambda=\{(\mathbf{x},y)|\mathbf{y}=\lambda\mathbf{x}\}.$$

We can think of X as corresponding to the "value" ∞. The axis spaces X and T_λ are then the solution spaces of the corresponding system of $t+1$ simultaneous linear equations in the $2t+2$ homogeneous coordinates $x_1,x_2,\ldots,x_{t+1}$; $y_1,y_2,\ldots,y_{t+1}$, where $\mathbf{x}=(x_1,\ldots,x_{t+1})$ and $\mathbf{y}=(y_1,\ldots,y_{t+1})$. Similarly the space $U=\Sigma_{t+1}$ is given by a system of t linear equations. Thus to find the intersection of an axis space with U, we solve a system of $2t+1$ linear equations in $2t+2$ unknowns. From the "Comment on Theorem 1.2" at the end of this paper we may set

$$x_1 + x_2 + \cdots + x_{t+1} + y_1 + y_2 + \cdots + y_{t+1} = 1, \quad \text{say}.$$

We now have a system of $2t+2$ linear equations in $2t+2$ unknowns. The hypothesis of Theorem 1.2 implies that the solution is unique. Thus we may solve, using Cramer's rule. To find the intersection $P(\lambda)$ of T_λ with U, let $A = A(\lambda)$ be the corresponding coefficient matrix of the system above. Then

$$P(\lambda) = \frac{1}{\det A(\lambda)}(p_1(\lambda), p_2(\lambda), \ldots, P_{2t+2}(\lambda)),$$

where $p_1(\lambda), p_2(\lambda), \ldots, p_{2t+2}(\lambda)$ are polynomials of degree at most $t+1$ in λ, and $\det A(\lambda) \neq 0$. Therefore we can take $P(\lambda)$ to have homogeneous coordinates $(p_1(\lambda), p_2(\lambda), \ldots, p_{2t+2}(\lambda))$. We now have a mapping f from $\mathrm{GF}(q)$ to the points $P(\lambda)$ defined by

$$f(\lambda) = (p_1(\lambda), p_2(\lambda), \ldots, p_{2t+2}(\lambda)).$$

Put

$$P(\infty) = f\left(\frac{1}{\lambda}\right)\bigg|_{\lambda=0}.$$

Using the above system of linear equations, it can be checked that $P(\lambda)$ is the intersection of the space $U = \Sigma_{t+1}$ with the axis space X of R_t. Thus we have a rational mapping f from the projective line $\mathrm{PG}(1,q)$ to the $q+1$ points of $C = \{P(\lambda), P(\infty)\}$. Therefore the points of C lie on a rational curve. ■

Remark. If q is odd, then any set of $q+1$ points in $\mathrm{PG}(3,q)$, with no four coplanar, must form the points of a normal rational curve in $\mathrm{PG}(3,q)$. However, this is not the case if q is even (see [13]). Thus *Theorem 1.2 yields a refinement of Corollary 1.1 in the case when q is even*. Our next result will be used in determining when the curve C in Theorem 1.2 is a *normal* rational curve in U.

Lemma 1.3. *In $\Gamma = \mathrm{PG}(2t-2,q)$ with $t \geq 3$, let $\Omega_1, \Omega_2, \ldots, \Omega_{q+1}$ be a collection of $q+1$ subspaces each having dimension $t-1$. Let J be a point of Γ such that any two distinct spaces Ω_i, Ω_j ($i \neq j$, $1 \leq i, j \leq q+1$) intersect in J and in no other point of Γ. Let B denote the set of all points of Γ that lie in* *some one of the spaces* Ω_i, $1 \leqslant i \leqslant q+1$. *Then there exists a* $(t-2)$*-dimensional subspace of* Γ *that contains no point of* B.

Proof. Let H be any $(2t-3)$-dimensional subspace of Γ not passing through J. Each space Ω_i meets H in a subspace H_i of H where H_i has dimension $t-2$. By the intersection property of the Ω_i these H_i form a partial $(t-2)$-dimensional spread in $H = \mathrm{PG}(2(t-2)+1, q)$. An incidence argument shows that this partial spread is not maximal if $t \geqslant 3$: indeed, for $q \geqslant 4$ this is an immediate consequence of Theorem 4 in [7]. ∎

The following theorem shows that in Theorem 1.2 the curve C need not be a *normal* rational curve in $U = \Sigma_{t+1}$ if $t > 2$. If $t = 2$ it must be normal, as follows from Corollary 1.1.

THEOREM 1.4. *In* $\Lambda = \mathrm{PG}(2t+1, q)$ *let* R_t *be a t-regulus,* $t \geqslant 3$. *Then there exists a* $(t+1)$*-dimensional subspace* U *of* Λ *and a nondegenerate conic* C *lying in a plane* π *of* U *such that the intersection of* R_t *with* U *is exactly the set of points on* C.

Proof. Denote the $q+1$ axis spaces of R_t by $\Sigma_1, \Sigma_2, \ldots, \Sigma_{q+1}$. Let d_1 be any line in Σ_1. As in Tyrrell and Semple [16], the transversals to R_t through the points of d_1 form a 1-regulus R_1. Each line of R_1 meets each axis space Σ_i in a unique point. Moreover (see [16, p. 13]) these $q+1$ points form a line in Σ_i. Let J_3 be the unique 3-dimensional space containing R_1. Let π be any plane of J_3 that is not tangent to R_1. Then each line of R_1 meets π in a unique point, and these points lie on a nondegenerate conic $C = \{P_1, P_2, \ldots, P_{q+1}\}$. From the above we have $P_i \in \Sigma_i$, $1 \leqslant i \leqslant q+1$. Let H_{t+1} be any $(t+1)$-dimensional subspace of $\Lambda = \mathrm{PG}(2t+1, q)$ that contains π. Suppose that H_{t+1} contains a further point Q of R_t with Q not on C. Then Q lies on some axis space of R_t, say $Q \in \Sigma_i$. Then all points of the line P_iQ lie in Σ_i. Then H_{t+1} also contains all points of the line joining P_i to Q. Let S denote the totality of all such lines P_iQ that join a point P_i of C to a point Q of Σ_i, $1 \leqslant i \leqslant q+1$, where Q is not a point of C. Each line l of S forms a 3-dimensional space $\langle l, \pi \rangle$ with π. Let B_3 denote the totality of all such 3-dimensional subspaces $\langle l, \pi \rangle$. It can be shown that if $l_1 \neq l_2$, then $\langle l_1, \pi \rangle \neq \langle l_2, \pi \rangle$, unless $\langle l_1, \pi \rangle = J_3 = \langle l_2, \pi \rangle$. It follows from this that the only 3-dimensional space in B_3 that contains both a line in Σ_i and a line in Σ_j (for some $i \neq j$, $1 \leqslant i, j \leqslant q+1$) is the distinguished 3-dimensional space J_3 containing the regulus R_1. Let us now work in the quotient projective geometry $\Gamma = \Lambda / \pi$. The points of Γ are all of the 3-dimensional subspaces of Λ that contain π. A $(t+1)$-dimensional subspace of Λ containing π yields a $(t-2)$-

dimensional subspace of Γ. The set B_3 in Λ yields a set B of points in Γ. Those lines l of S that lie in Σ_i, $1 \leq i \leq q+1$, yield the set of points in a $(t-1)$-dimensional subspace Ω_i of Γ. Moreover any two such subspaces Ω_i and Ω_j with $i \neq j$ intersect in the point J of Γ and in no other point of Γ. Here J corresponds to the subspace J_3 of Γ. The result now follows from Lemma 1.3. ■

2. ON 2-SPREADS OF PG$(5, q)$, $q \neq 2$

Let W be a partial 2-spread of PG$(5, q)$. We say that the plane π is a *transversal plane* of W if π satisfies the following two conditions:

(1) π intersects each plane of W in at least a point.
(2) Each point of π is contained in some plane of W.

LEMMA 2.1. *Let W be a partial 2-spread of* PG$(5, q)$, $q \neq 2$. *Assume that $\pi(1) \in W$, and that π is a transversal plane of W which intersects $\pi(1)$ in a line. Suppose that δ is another transversal plane of W which has no point in common with π. Then*

(i) *δ meets exactly one element $\pi(0)$ of W in a line, and $\pi(0) \neq \pi(1)$;*
(ii) *each 2-regulus contained in W must contain the planes $\pi(0)$ and $\pi(1)$.*

Proof. Since π is a transversal plane of W, and since π meets an element of W in a line we get $|W| = q^2 + 1$. Thus any other transversal plane δ of W has to meet some element $\pi(0)$ of W in a line. Because δ is disjoint from π, $\pi(0) \neq \pi(1)$. To prove (ii), let δ meet $\pi(1)$ in a point P and let Σ_3 be the 3-space generated by P and $\pi(0)$. Note that Σ_3 contains δ. Let $R_2 \subset W$ be a 2-regulus that does not have $\pi(0)$ as an axis plane. Since R_2 and Σ_3 lie in a 5-dimensional projective space, Σ_3 meets each axis plane of R_2 in at least a point. No axis plane σ of R_2 can meet Σ_3 in a line. For if it did, this line would intersect $\pi(0)$, since $\pi(0)$ lies in Σ_3. Then σ intersects $\pi(0)$. Now $\sigma \neq \pi(0)$, since $\pi(0) \notin R_2$. Then two distinct planes $\sigma, \pi(0)$ of W intersect. But this is impossible. Thus Σ_3 intersects R_2 in a set of $q+1$ points which, by Corollary 1.1, either lie on a transversal line of R_2 or have the property that no four are coplanar. First suppose that the points lie on a transversal line d of R_2. Since $\pi(0)$ is a plane of Σ_3, d must meet $\pi(0)$ in a point. But this is impossible, since each point of d is covered by a unique plane in R_2, and $\pi(0)$ is not in R_2. Next, suppose the $q+1$ points given by $\Sigma_3 \cap R_2$ have the property that no four are coplanar. Now δ lies in Σ_3, and δ, being a transversal plane of W, meets all planes of W. In particular δ meets all planes of R_2. Thus all the $q+1$ points of $\Sigma_3 \cap R_2$ also lie in the plane of δ. This is impossible if $q \geq 3$, since $q+1 > 3$ if $q \geq 3$. We conclude that any regulus

$R_2 \subset W$ must contain $\pi(0)$. By reversing the roles of π and δ we see that R_2 must also contain $\pi(1)$. ■

As in [3], a 2-spread S is said to be *regular* if each plane of the 2-regulus determined by any three planes of S is also contained in S. This is equivalent to demanding that if d is any line that does not lie in any plane of S, then the $q+1$ planes of S that pass through the $q+1$ points of d must form a 2-regulus.

THEOREM 2.2. *Let S be a regular spread of* PG(5, q) *with* $q \neq 2$. *Let* $\pi(1)$ *be in* S, *and let* π *be a plane meeting* $\pi(1)$ *in a line. Denote by* $S(\pi)$ *the set of all planes of* S *that intersect* π *in at least a point. Put* $W = (S \setminus S(\pi)) \cup \{\pi\}$. *Then* W *is a maximal partial spread with* $|W| = q^3 - q^2 + 1$.

Proof. Suppose that δ is a plane that is disjoint from each plane of W. Then δ is a transversal plane of $S(\pi)$. Therefore, as in Lemma 2.1, δ meets some plane $\pi(0)$ in a line, with $\pi(0) \neq \pi(1)$. Now we can obtain a contradiction to Lemma 2.1 by exhibiting a 2-regulus R_2 contained in $S(\pi)$ and having the property that R_2 contains, say, $\pi(1)$ but not $\pi(0)$. To do this, let f be the line of intersection of π and $\pi(1)$. Let $\pi \cap \pi(0) = Q$. Let d be any line of π not on Q with $d \neq f$. Then since S is regular, the $q+1$ planes of $S(\pi)$ that contain the $q+1$ points of d form a regulus R_2 that has d as a transversal line. Since d meets f, R_2 contains $\pi(1)$. Moreover, $\pi(0) \notin R_2$, since $\pi(0)$ does not intersect d. ■

3. ON PARTIAL 2-SPREADS OF PG(5, 2)

In Section 2 the assumption that $q \neq 2$ was used in the proof of Lemma 2.1 and thus in the proof of Theorem 2.2. In fact Theorem 2.2 is true for all prime-power values of q: the case $q = 2$ is treated in this section. Some of our results here can be shown by standard methods of spreads (as in Bruck and Bose [3], or the paper [16]) and indicator sets (as in Sherk [16]). However it will be convenient and instructive to use a linear representation of PG(2, q^2) in the space PG(5, q). A summary of this representation is given here. Further details are in [2, 11]. Consider the lattice of subfields of GF(q^6) over GF(q). Let θ be a primitive element of GF(q^6). Setting

$$\beta = \theta^{q^4+q^2+1}, \tag{i}$$

we have that β is a primitive element of GF(q^2). Now GF(q^6) is a

3-dimensional vector space over GF(q^2): we denote this vector space by $V(q^2)$. Note that $V(q^2)$ may also be thought of as a 6-dimensional vector space over GF(q), whose 1-dimensional subspaces form the points of PG($5, q$). The points and lines of PG($2, q^2$) may be taken as the 1- and 2-dimensional subspaces of $V(q^2)$, respectively. The set of points of PG($2, q^2$), when viewed over GF(q), yields a collection L_1 of projective lines that forms a 1-spread of PG($5, q$). The set of lines of PG($2, q^2$), when viewed over GF(q), is represented by a collection, H_3, of projective 3-spaces of PG($5, q$). This collection is a dual spread in the sense that each hyperplane of PG($5, q$) contains exactly one 3-space belonging to H_3. Each element Ω_3 of H_3 contains a subset of L_1 that forms a regular 1-spread of Ω_3.

For a point P in PG($5, q$) the mapping

$$P \to \beta^i P \tag{ii}$$

(i fixed, $1 \leqslant i \leqslant q$) is a collineation of PG($5, q$) leaving fixed each line of L_1 and permuting the points of each line in L_1.

A plane of PG($5, q$) is said to be of type 1 (type 2) if it contains (does not contain) an element of L_1. In this representation, each Baer subplane of PG($2, q^2$) is represented by a unique 2-regulus R. Each axis plane of R is of type 2, and each transversal line of R is an element of L_1 [2, p. 390].

A particular regular 2-spread P_2 of PG($5, q$) is described now. Setting

$$\delta = \theta^{q^3+1}, \tag{iii}$$

δ is a primitive element of GF(q^3). Now GF(q^6) yields a two-dimensional vector space over GF(q^3), which we denote by $V(q^3)$. Then $V(q^3)$ gives a representation of PG($1, q^3$) as follows. The q^3+1 points of PG($1, q^3$) have a vector representation of the form

$$(1, \alpha) \text{ or } (0,1), \qquad \alpha \in \mathrm{GF}(q^3). \tag{iv}$$

When viewed over GF(q), the points of PG($1, q^3$) determine a collection of planes of PG($5, q$) that forms a regular 2-spread, P_2, of PG($5, q$). Further, each plane belonging to P_2 is of type 2 with respect to the spread L_1. The spread P_2 is invariant under the collineations induced by the mappings in (ii). Under powers of β, the planes belonging to P_2 are partitioned into a collection B of $q^2 - q + 1$ 2-reguli, each representing a Baer subplane of PG($2, q^2$). This collection of subplanes partitions the points and lines of PG($2, q^2$). Call B a *Baer subplane partition* of PG($2, q^2$).

LEMMA 3.1. *Let π_β be a Baer subplane of* PG$(2, q^2)$, *and let R_2 be the corresponding 2-regulus of* PG$(5, q)$ *in the previous representation. Let π be a plane of* PG$(5, q)$ *having no point in common with any axis plane of R_2. Then π is of type 2 and is an axis plane of a 2-regulus representing a Baer subplane that has no point or line in common with π_β.*

Proof. Suppose π were of type 1. The q^2+1 elements of L_1 meeting π in a point or a line would be in a unique 3-space Ω_3 belonging to H_3. Now Ω_3 represents a line of PG$(2, q^2)$, which would therefore intersect any Baer subplane in at least a point of that subplane. Then Ω_3 would intersect the 2-regulus R_2 in at least a transversal line p. Since p and π would be in the 3-space Ω_3, they would have a point in common. Thus π would intersect some axis plane of R_2 in a point, which is a contradiction. Therefore π is of type 2. The images of π under the collineations given in (ii) are the axis planes of a 2-regulus representing a Baer subplane having no point or line in common with π_β. ■

LEMMA 3.2. *Given a set $\{\sigma_i\}$, $1 \leqslant i \leqslant 4$, of four mutually disjoint planes in* PG(5, 2), *there is a regular 2-spread containing them.*

Proof. For the regular spread P_2 described earlier, let $q = 2$. We have

$$P_2 = \bigcup_k \{\pi(k, j)\}, \qquad 1 \leqslant k, j \leqslant 3, \tag{v}$$

where, for each k, the three planes $\pi(k, j)$ form a 2-regulus that represents one of the three Baer subplanes in a subplane partition of PG(2, 4). Without loss of generality, the 2-regulus determined by σ_1, σ_2, σ_3 may be taken as $\{\pi(1, j)\} = R_2$. Since σ_4 is a plane disjoint from R_2, we have from Lemma 3.1 that σ_4 is of type 2. Hence it is an axis plane of a unique 2-regulus, R_2^*, representing a Baer subplane of PG(2, 4). The remaining points and lines of PG(2, 4) not in the subplanes determined by R_2 or R_2^* form a subplane which is necessarily represented by a 2-regulus, say $R_2^\#$. The axis-planes of R_2, R_2^*, and $R_2^\#$ form a regular 2-spread containing the four given planes. ■

THEOREM 3.3. *Let S be a 2-spread of* PG(5, 2) *with $\pi(0) \in S$, and let π be a plane so that $\pi \cap \pi(0)$ is a line w. Denote by $S(\pi)$ the set of all planes of S that intersect π. Then there is no plane σ that is a transversal plane to $S(\pi)$, and that is also skew to π.*

Proof. Since all regular 2-spreads of PG$(5, q)$ are equivalent under the collineation group of PG$(5, q)$, and since all spreads of PG(5, 2) are regular, the 2-spread S may be taken to be the 2-spread P_2 described earlier. The group of P_2 being 3-transitive on the planes in P_2 implies that of the four planes of P_2 not in $S(\pi)$ three may be taken as $\pi(1,j)$, $1 \leqslant j \leqslant 3$. By relabeling, the fourth may be taken to be $\pi(2,1)$. As before, put $R_2 = \{\pi(1,j)\}$, $1 \leqslant j \leqslant 3$. Since π is disjoint from the 2-regulus R_2, it follows from Lemma 3.1 that π is of type 2. Suppose that $\pi(0)$ is an axis plane of the 2-regulus $\{\pi(3,j)\}$. Then π meets one of the axis planes of $\{\pi(3,j)\}$ namely $\pi(0)$, in the line w. Also, π must intersect the other two axis planes of $\{\pi(3,j)\}$ in a point each. Let d be the line joining those two points. Then d meets w, so d contains $q+1=3$ points of $\{\pi(3,j)\}$. Since d is in π, it follows that π contains a transversal line d of the 2-regulus $\{\pi(3,j)\}$. This means that π is of type 1, which is impossible. Therefore $\pi(0)$ must be an element of the 2-regulus $\{\pi(2,j)\}$, say $\pi(0)=\pi(2,2)$. Let σ be a transversal plane of $S(\pi)$ with σ disjoint from π. Then σ must meet $\pi(2,2)$ in a point not on w. Also, arguing in a manner similar to the above, it follows that σ must meet $\pi(2,3)$ in a line v. Let $X=\pi \cap \pi(2,3)$. Note that X is not on w, as $\pi(2,3)$ is skew to $\pi(2,2)$. Since $\pi(2,1)$ is skew to π, the transversal line of $\pi(2,j)$ on X cannot lie in π: this line must therefore meet $\pi(2,2)$ in a point Y, where Y is not on w. Since σ is skew to π, v cannot contain X, so there are at most four possibilities for v in $\pi(2,3)$. Put $Z=\sigma \cap \pi(2,2)$. Now σ is skew to π, so Z is off w. Also, Z is not on v, since $\pi(2,2)$ is skew to $\pi(2,3)$. Again, the transversal line b of $\{\pi(2,j)\}$ on Z cannot lie in σ, so that v cannot contain the point $T=b \cap \pi(2,3)$. We consider now two cases:

(1) $Z=Y$.
(2) $Z \neq Y$.

Choose X and w. For case (1) there are at most four choices for v. This leaves at most $1\times 4=4$ choices for $\sigma=\langle Z, v\rangle$. We know that neither X nor T lies on v. In case (2) we have $X \neq T$. Thus, for case (2), there are at most two choices for v, and so at most $3\times 2=6$ choices for $\sigma=\langle Z, v\rangle$. Therefore there are at most ten possibilities for σ. By a hand calculation it can be shown that none of these ten possibilities yields a transversal plane σ of $S(\pi)$ such that σ is also skew to π. The proof of Theorem 3.3 is now complete. ∎

Corollary 3.4. *There is a maximal partial* 2*-spread* W *of* PG(5, 2) *with* $|W|=5$.

Proof. The set of planes W, which is described in Theorem 2.2 for the case of $q \neq 2$, yields also a partial 2-spread W of PG(5, 2) with $|W|=5$. It follows from Theorem 3.3 that W is also a maximal partial spread. ∎

Corollary 3.5. *There is a unique* 2*-spread of* PG(5, 2) *containing a set of four pairwise disjoint planes.*

Proof. Denote the four planes by $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. By Lemma 3.2 there is a regular 2-spread S containing them. Let $W \neq S$ be another 2-spread containing them. Let $\pi \in W$, $\pi \notin S$. Now π is skew to $\{\sigma_i\}$, $1 \leqslant i \leqslant 4$. Since S is a spread, each point of π is covered by an element of S. Thus π meets one of the remaining five planes of S, say $\pi(0)$, in a line, and the four remaining planes of S, say $\sigma_6, \sigma_7, \sigma_8, \sigma_9$ in a point each. In the notation of Theorem 3.3 we have $S(\pi) = \{\pi(0), \sigma_6, \sigma_7, \sigma_8, \sigma_9\}$. Now any plane σ of W different from π and the $\{\sigma_i\}$, $1 \leqslant i \leqslant 4$, must be skew to π and the $\{\sigma_i\}$, $1 \leqslant i \leqslant 4$. Thus σ must be a transversal plane to $S(\pi)$ and must also be skew to π. But this contradicts Theorem 3.3 ■

Maximal partial t-spreads of $PG(2t+1, q)$ are studied in [7]. Bounds on the size of such partial spreads are obtained under the restriction that $q \geqslant 4$. The following complements this work in [7].

THEOREM 3.6. *Let W be a maximal partial 2-spread of* PG(5, 2). *Then* $|W| = 5$. *Moreover, this case occurs by Corollary* 3.1.

Proof. From Corollary 3.5, we have $|W| \geqslant 5$. Choose any four planes in W and denote them by $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. As in Lemma 3.2 there is a regular 2-spread S containing them. Now since W is not a spread we have $W \not\subseteq S$. Then W must contain some plane π which is not in S. Then, as in the proof of Corollary 3.5, π meets some plane $\pi(0)$ of S in a line. As in the proof of Corollary 3.5, and using the notation there, it follows that any other plane σ of W with $\sigma \notin \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \pi\}$ must be skew to π and must be a transversal plane to $S(\pi)$. An appeal to Theorem 3.3 completes the proof. ■

In [8, p. 147] Drake asks if there is a set M of four mutually orthogonal latin squares of order 8 having the property that M is not properly contained in a larger set of m.o.l.s. of order 8. Such a set would define a net of order 8 and degree 6 that would be maximal. Theorem 3.3 may be restated to give a partial answer to Drake's question. We refer to [6] and [15] for a description of certain kinds of nets called *translation nets*. The maximal partial spreads of Theorem 2.2 and Corollary 3.4 immediately yield certain translation nets of order q^3. It would be of interest to determine whether or not such translation nets are also maximal nets. In [5] and [6, p. 248] it is shown that certain kinds of maximal partial spreads in $PG(3, q)$ yield maximal nets.

In what follows, $V(6, q)$ means a vector space of dimension 6 over $GF(q)$.

COROLLARY 3.7. *Suppose there exists a maximal set of four mutually orthogonal latin squares of order* 8. *Let N be the associated net of order* 8 *and degree* 6. *Then N is not isomorphic to any maximal translation net M defined* *by 3-dimensional subspaces in* $V(6, 2)$ (*and having degree* 6 *and order* $2^3 = 8$).

Proof. Suppose that N is isomorphic to M. Now M yields a partial 2-spread W of PG(5, 2) with $|W| = 6$. By Theorem 3.6, W is contained in a larger partial spread W' of PG(5, 2) with $|W'| = 7$. W' then yields a net M' of degree 7 and order 8 that properly extends M. M' then gives a set of five m.o.l.s. that properly extends the original maximal set of four m.o.l.s. This is a contradiction. Thus N is not isomorphic to M. ■

Comment on Theorem 1.2. The number of points in the set $C = \{P(\lambda), P(\infty)\}$ is $q+1$. Assume that C is not a line in U. Then an easy counting argument shows that there exists a hyperplane A_t of dimension t in U such that no point of C lies on A_t. Now take any hyperplane $B = B_{2t}$ of dimension $2t$ in $\mathrm{PG}(2t+1, q)$ containing A_t such that B does not contain U. By adjusting coordinates we may assume that the equation of B is

$$x_1 + x_2 + \cdots + x_{t+1} + y_1 + y_2 + \cdots + y_{t+1} = 0.$$

As stated in the proof of Theorem 1.2, we may now demand that

$$x_1 + x_2 + \cdots + x_{t+1} + y_1 + y_2 + \cdots + y_{t+1} = 1.$$

REFERENCES

1 A. Beutelspacher, Partial spreads in finite projective spaces and partial designs, *Math. Z.* 145:211–229 (1975).
2 R. C. Bose, On a representation of the Baer subplanes of the Desarguesian plane $\mathrm{PG}(2, q^2)$ in a projective five dimensional space $\mathrm{PG}(5, q)$, in *Proceedings of the Conference on Combinatorial Theory #17*, Rome, 3–15 Sept. 1973, pp. 381–391.
3 R. H. Bruck and R. C. Bose, Linear representations of projective planes in projective spaces, *J. Algebra* 4:117–172 (1966).
4 A. A. Bruen, Partial spreads and replaceable nets, *Canad. J. Math.* 23:381–391 (1971).
5 ______, Unimbeddable nets of small deficiency, *Pacific J. Math.* 43:51–54 (1972).
6 ______, Collineations and extensions of translation nets, *Math. Z.* 145:243–249 (1975).
7 ______, Blocking sets and skew subspaces of projective space, *Canad. J. Math.* 23:628–630 (1980).
8 D. A. Drake, Maximal sets of latin squares and partial transversals, *J. Statist. Plann. Inference* 1:143–149 (1977).

9 G. L. Ebert, Maximal strictly partial spreads, *Canad. J. Math* 30:483–489 (1978).
10 ______, A new lower bound for msp spreads, in *Proceedings of the 10th Conference on Combinatorics, Graph Theory and Computing*, 2–6 Apr. 1979, Florida Atlantic Univ., to appear.
11 J. W. Freeman, A representation of the Baer subplanes of PG(2, q^2) in PG(5, q) and properties of a regular spread PG(5, q), in *Proceedings of the International Conference on Projective Planes*, Washington State U.P., 1973, pp. 91–97.
12 ______, On the Grassmanian of lines in PG(4, q) and R(1, 2) reguli, *J. Geom.* 5(2):159–183 (1974).
13 J. W. P. Hirschfeld, Rational curves on quadrics over finite fields of characteristic two, *Rend. Mat. (4), Ser. Vi*, 3:772–795 (1971).
14 D. M. Mesner, Set of disjoint lines in PG(3, q), *Canad. J. Math.* 19:273–280 (1967).
15 F. A. Sherk, Indicator sets in an affine space of any dimension, *Canad. J. Math.* 21:211–224 (1979).
16 J. A. Tyrrell and J. B. Semple, *Generalized Clifford Parallelism*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 61, Cambridge U.P., 1971.

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